Journal of Statistical Physics, Vol. 83, Nos. 5/6, 1996

Bounds on Enhanced Turbulent Flame Speeds for Combustion with Fractal Velocity Fields

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Received October 3, 1995

Rigorous upper bounds are derived for large-scale turbulent flame speeds in a prototypical model problem. This model problem consists of a reaction-diffusion equation with KPP chemistry with random advection consisting of a turbulent unidirectional shear flow. When this velocity field is fractal with a Hurst exponent H with $0 < H < 1$, the almost sure upper bounds suggest that there is an accelerating large-scale turbulent flame front with the enhanced anomalous propagation law $y = C_H t^{1+H}$ for large renormalized times. In contrast, a similar rigorous almost sure upper bound for velocity fields with finite energy yields the turbulent flame propagation law $y = \tilde{C}_H t$ within logarithmic corrections. Furthermore, rigorous theorems are developed here which show that upper bounds for turbulent flame speeds with fractal velocity fields are not self-averaging, i.e., bounds for the ensemble-averaged turbulent flame speed can be extremely pessimistic and misleading when compared with the bounds for every realization.

KEY WORDS: Turbulence; combustion; fractal fields.

INTRODUCTION

A significant experimental observation in premixed turbulent combustion is enhanced flame propagation velocities exceeding the ordinary laminar flame speed.[1,2] In turbulent premixed combustion, the convecting fluid velocity typically involves many spatiotemporal scales which are all larger than the flame thickness. An important research topic in the combustion community[3-14] involves developing theories which provide effective

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0022-4715/96/0600-0933$09.50/0

enhanced flame speeds at large scales in turbulent combustion and exploit the great disparity between the largest scales of variation of the velocity field and the flame thickness.[9-11] Some of the approaches developed in an ad hoc fashion to predict large-scale turbulent flame speeds include the renormalization group[7,8] as well as other closure theories.[5,6] Recently, the authors[12] have developed a rigorous renormalization theory for large-scale front dynamics for turbulent reaction-diffusion equations with two separate velocity scales which are both larger than the flame thickness. Embid *et al.*[13] have developed the implications of this theory in a prototype example and have compared the rigorous theory with several ad hoc approximation procedures mentioned earlier in this context of turbulent velocity fields with two separate scales.[14]

Here we begin the rigorous mathematical study of turbulent reaction-diffusion equations with many energetic velocity scales and obtain rigorous upper bounds on the turbulent flame speed in a prototypical model problem (see p. 3 of ref. 12). These rigorous upper bounds exhibit a change of phase from bounded turbulent flame velocities on large scales, when the velocity statistics have a band-limited energy spectrum, to anomalous temporal evolving and accelerating turbulent flame velocities, when the velocity statistics have fractal scaling over a wide spatial range with interesting dependence on the Hurst exponent H, which serves as the phase parameter. For the Kolmogoroff velocity spectrum in the model with $H=1/3$, these formulas are qualitatively similar to those developed by Kerstein and Ashurst[5] through a completely different approach. Furthermore, we also develop rigorous theorems which show that upper bounds for turbulent flame speeds with fractal velocity fields are not self-averaging, i.e., bounds for the ensemble-averaged turbulent flame speed can be extremely pessimistic and misleading when compared with the bounds for almost every realization. This phenomenon is a rigorous signature of intermittency in turbulent combustion with fractal velocity fields where exceptional sets of rare events contribute disproportionately to the ensemble average when compared with the typical case. The rigorous bounds established here also indicate that such phenomena do not occur when the velocity field has a band-limited energy spectrum and there can be at most much weaker intermittency.

In this paper we study the model problem for turbulent combustion given by

$$\begin{cases} T_t - \frac{\kappa}{2} \Delta T + v_\lambda(x)\, T_y + w T_x + KT(T-1) = 0 & \text{in } R^2 \times (0, \infty) \\ T|_{t=0} = T_0 & \text{on } R^2 \times \{0\} \end{cases} \tag{1}$$

with the diffusion coefficient $\kappa > 0$, the reaction rate K, and the sweeping velocity w all fixed constants, and with the initial data $T_0(x, y)$ given by

$$T_0(x, y) = \Theta(y) = \begin{cases} 1 & \text{if} \quad y < 0 \\ 0 & \text{if} \quad y > 0 \end{cases} \tag{2}$$

The velocity field $v_\lambda(x)$ is a stationary zero-mean Gaussian random field with the spectral representation

$$v_\lambda(x) = (2\pi)^{-1/2}\, V_0 \int_{|k| \geqslant \lambda} e^{ixk} \psi^{1/2}(|k|)\, |k|^{-(1/2) - H}\, W(dk) \tag{3}$$

and the parameter H satisfies

$$-\infty < H < 1 \tag{4}$$

In (3), $W(dk)$ is complex Gaussian white noise with

$$\langle W(dk)\, W(dk') \rangle = \delta(k + k')\, dk\, dk'$$

Here and below, $\langle \cdot \rangle$ denotes averaging with respect to the velocity statistics. The ultraviolet cutoff $\psi_\infty(|k|)$ is a nonnegative, continuous, rapidly decreasing function satisfying

$$\psi_\infty(0) = 1 \tag{5}$$

while the infrared cutoff parameter λ satisfies $\lambda \ll 1$.

The prototype problem in (1) combines the fully developed turbulent shear flow models from refs. 15 and 16 with KPP chemistry. The problem in (1)–(4) arises naturally through nondimensionalizing the turbulent reaction-diffusion equations by utilizing velocity dissipation scales for space and time from conventional turbulence theory.[15] The fact that the cutoff function $\psi_\infty(|k|)$ is rapidly decreasing automatically guarantees that the velocity field has very little energy below the dissipation scale and the assumptions that κ and K both remain finite with this choice of scales guarantee that the laminar flame thickness $(\kappa/K)^{1/2}$ is of the order of the dissipation length scale. The parameter λ represents the ratio of the dissipation scale to the integral scale[15] and in conventional turbulence theory, $\lambda = (\mathrm{Re})^{-3/4}$, with Re the Reynolds number.[15] Here we are especially interested in the situation where there is a fully developed turbulent flow field with many spatial scales which influence the large-scale turbulent flame front propagation; thus, we are interested in the behavior of (1) as $\lambda \to 0$.

The character of the velocity field $v_\lambda(x)$ in the limit $\lambda \to 0$ is completely different depending on whether the parameter H satisfies $H<0$ or $0<H<1$. For $H<0$, the mean energy $\lim_{\lambda\to 0}\langle v_\lambda^2(0)\rangle$ is finite; for $0<H<1$ there is infrared divergence of energy and instead the velocity difference satisfies for $|x| \gg 1$ and $0<H<1$

$$\lim_{\lambda \to 0} \langle (v_\lambda(x+x') - v_\lambda(x'))^2 \rangle = C_H^2 V_0^2 |x|^{2H} \tag{6}$$

with C_H a universal constant. The identity in (6) guarantees that, when viewed at large scales, the Gaussian random velocity field is nowhere differentiable and fractal with a Hurst exponent H for $0<H<1$. We expect an anomalous boost in turbulent flame speeds in the parameter regime $0<H<1$ due to the wrinkling at all scales of the flame front by the random shear flow, which enhances combustion. On the other hand, for $H<0$, the velocity field has finite energy and we anticipate that the enhanced flame propagation behaves similar to that in the situation with separate velocity scales.[12–14] The bounds which we state below and derive in precise form in the remainder of this paper confirm all of this intuition.

To study the turbulent flame speeds at large scales and long times, we introduce the rescaled variable

$$T^\lambda(x, y, t) = T(\lambda^{-1}x, \alpha(\lambda)^{-1} y, \beta(\lambda)^{-1} t) \tag{7}$$

where the scaling group parameters $\alpha(\lambda)$, $\beta(\lambda)$ are chosen to give a nontrivial limit. For the conventional KPP problem with separate scales, the choices $\alpha(\lambda) = \lambda$, $\beta(\lambda) = \lambda$ yield large-scale geometric front propagation.[12, 17, 18] In general we need to choose $\alpha(\lambda)$ and $\beta(\lambda)$ as functions of the parameter H which are compatible with the intrinsic scaling of (1) in the large-scale limit.[19] The next results indicate that this can be achieved for almost sure upper bounds on the turbulent flame speed in the y direction:

Theorem 1 (The fractal case). Consider the regime $0<H<1$ with fractal large-scale velocity fields and choose the anomalous scaling $\alpha(\lambda) = \lambda^{1+H}$ and $\beta(\lambda) = \lambda$ in (7). There is a universal constant $\tilde{C}_H$, depending on $|x| + |t|$ and C_H in (6), such that as $\lambda \to 0$, $T^\lambda(x, y, t) \to 0$ a.s. for large t provided y satisfies the bound

$$y > V_0 \tilde{C}_H t((2\kappa K)^{1/2} t)^H \tag{8}$$

Theorem 2 (The smooth case). Assume H satisfies $H<0$, choose $\alpha(\lambda) = \lambda \ln \lambda|^{1/2}$ and $\beta(\lambda) = \lambda$, and let $\gamma_0 = V_0((2\pi)^{-1} \int \psi_\infty(|k|)\,|k|^{-1-2H}\,dk)^{1/2}$. Then for almost every realization of the velocity field, there is a constant

C depending on that realization such that as $\lambda \to 0$, $T^{\lambda}(x, y, t) \to 0$ for large t provided that y satisfies the bound

$$y > C\gamma_0(\ln(|x| + t))^{1/2}\, t \tag{9}$$

A precise version of Theorems 1 and 2 applying at all rescaled times t is stated and proved, respectively, in Sections 2 and 3 of this paper.

The upper bound in (8) suggests that for fractal velocity fields, turbulent flame fronts for (1) in this scaling regime accelerate with the anomalous enhanced propagation law

$$y = V_0((2\kappa K)^{1/2})^H\, \tilde{C}_H t^{1+H} \qquad \text{for} \quad 0 < H < 1 \tag{10}$$

while with finite energy for the large-scale velocity the propagation rate is

$$y = C_*(H)\, t \qquad \text{for} \quad H < 0 \tag{11}$$

with $(\ln t)^{1/2}$ corrections. Thus, the upper bounds on turbulent flame propagation in Theorems 1 and 2 predict a change of phases across the boundary $H = 0$ with the typical behavior from (11) for $H < 0$ and the anomalous behavior in (10) for the fractal regime with $0 < H < 1$. We note that the anomalous scaling law in (8) or (10) is scale invariant under the group of transformations.

$$y' = \lambda^{1+H} y, \qquad t' = \lambda t$$

utilized in rescaling T^{λ} in (7) and furthermore, the constant $V_0((2\kappa K)^{1/2})^H\, \tilde{C}_H$ has dimensional units compatible with this law. With the value of the Kolmogoroff exponent $H = 1/3$, Theorem 1 suggests that turbulent flames in this large-scale limit propagate according to a $t^{4/3}$ law which is reminiscent of a recent prediction by Kerstein and Ashurst.[5]

In Section 4, we derive upper bounds for the ensemble average over all velocity realizations,

$$\langle T^{\lambda} \rangle = \langle T(\lambda^{-1}x, \alpha(\lambda)^{-1}\, y, \beta(\lambda)^{-1}\, t) \rangle \tag{12}$$

We establish the following result:

Theorem 3 (Ensemble-averaged bounds). (A) For H with $0 < H < 1$, choose $\alpha(\lambda)$, $\beta(\lambda)$ to satisfy $\beta(\lambda) = \alpha^{2/3}(\lambda)\, \lambda^{-2H/3}$ and $\lambda\beta^{-1}(\lambda) \to 0$ as $\lambda \to 0$. Then as $\lambda \to 0$, $\langle T^{\lambda} \rangle \to 0$ for

$$y > (2K\kappa C_0)^{1/2}\, t^{3/2} \tag{13}$$

with $C_0 = V_0^2 \int_{|k| \geq 1} |k|^{1-\varepsilon}\, dk$.

(B) For H with $H<0$, choose $\alpha(\lambda)=\beta(\lambda)=\lambda$. Then as $\lambda\to 0$, $\langle T^{\lambda}\rangle \to 0$ for

$$y>(2K\kappa)^{1/2}\, t \tag{14}$$

By comparing (8) or (10) with (13), we observe that ensemble-averaged upper bounds are completely misleading for the fractal velocity regime with $0<H<1$ and exhibit no dependence on H. On the other hand, in the case $H<0$ with smooth velocity fields of finite energy, there is weak intermittency and the bounds in (9) and (11) are compatible with the ensemble-averaged bound in (14) within logarithmic corrections. Similar ensemble-averaged upper bounds as in Theorem 3 for the model from p. 3 of ref. 12 have been developed independently by Fedotov[20] in the special case $\alpha(\lambda)=\lambda$.

The only nonsteady feature of the velocity field discussed in this paper is through the large-scale sweeping effects of the mean flow w, which introduces streamlines which block the transport in the y direction. An important and interesting direction of research, even in the context of this model, is to incorporate more complex spatiotemporal statistics in the shear flow[15, 16] besides the large-scale sweep. Many of the tools developed here are relevant for that more complex and physically relevant situation, which is a topic of current investigation by the authors.

1. ALMOST SURE UPPER BOUNDS FOR FRACTAL VELOCITY FIELDS

Throughout this section we assume that

$$0<H<1 \tag{15}$$

and use the Gaussian random field V with spectral representation

$$V(x)=(2\pi)^{-1/2}\, V_0 \int_{|k|\geq 1} e^{ixk}\, |k|^{-(1/2)-H}\, W(dk) \tag{16}$$

In view of (15) the covariance

$$\Gamma(x)=\langle V(x)\, V(0)\rangle=(2\pi)^{-1}\, V_0 \int_{|k|\geq 1} e^{ixk}\, |k|^{-1-2H}\, dk \tag{17}$$

is finite, hence V is well defined.

The precise statement of Theorem 1 from the Introduction is:

Theorem 1 (The fractal case). Assume $0<H<1$ and choose $\alpha(\lambda)=\lambda^{1+H}$ and $\beta(\lambda)=\lambda$ in (7). For each $\eta>0$, $\zeta>0$, and $\delta\in(0, H)$ and almost every ω there exists $g_0 = g_0(\eta, \delta, \omega) > 0$ and $\tilde{C} = \tilde{C}(C_H, |w|, |x|)$ such that, as $\lambda \to 0$, $T^\lambda(x, y, t, \omega) \to 0$, provided

$$y > V_0 \tilde{C}_H t^{1+H} + 2g_0^{\delta - H} \, \|V\|_{L^\infty(-R, R)} \, (2(1+\zeta)\, \kappa K)^{(\delta - H)/2}\, t + \int_0^t V(x - ws)\, ds \tag{18}$$

with C_H as in (6), V is the Gaussian random field given by (16), and $R = |x| + (|w| + (2(1+\zeta)\, \kappa K)^{1/2})\, t$.

We continue with the following important remark.

Remark. The bound on y in (18) yields for $t \gg 1$ the bound claimed in (8). Indeed, since for $\rho \gg 1$, $\max_{|y| \leqslant \rho} V(y) \approx (2\Gamma(0) \ln \rho)^{1/2}$ and $\max_{|y| \leqslant \rho} |V(y)| \approx (2\Gamma(0) \ln \rho)^{1/2}$ (see Proposition A.1 in the appendix and refs. 21, 22, it follows that the contribution of $\|V\|_{L^\infty(-R, R)}$ for large t is of order

$$(2\Gamma(0) \ln[\,|x| + (|w| + (2(1+\zeta)\, \kappa K)^{1/2})\, t\,])^{1/2}$$

Similarly the contribution of $\int_0^t V(x - ws)\, ds$, which, incidentally, has the obvious physical interpretation as the Lagrangian path following the shear flow for the rescaled problem derived below in (24), is of order $t[2\Gamma(0)(\ln(|x| + |w|\, t))]^{1/2}$. Actually, since $\langle V \rangle = 0$, for the case with $w \neq 0$, the ergodic theorem yields that the contribution of $\int_0^t V(x - ws)\, ds$ is even more negligible for large t, since $t^{-1} \int_0^t V(x - ws)\, ds \to 0$ as $t \to \infty$. In view of all the above discussion, it is now clear that for large t the formula in (18) yields the claim in (8) from the Introduction.

We continue with a discussion which motivates the choice of the scaling in the theorem and explains the strategy of the proof.

The definition (7) of T^λ together with (1) and (2) and a simple calculation yield that T^λ solves

$$\begin{cases} \text{(i)} \quad T^\lambda_t - \dfrac{\kappa}{2\beta(\lambda)} (\lambda^2 T^\lambda_{xx} + \alpha^2(\lambda)\, T^\lambda_{yy}) + \dfrac{\alpha(\lambda)}{\beta(\lambda)} v_\lambda \left(\dfrac{x}{\lambda}\right) T^\lambda_y \\ \qquad + \dfrac{\lambda}{\beta(\lambda)} w T^\lambda_x + \dfrac{K}{\beta(\lambda)} T^\lambda (T^\lambda - 1) = 0 \qquad \text{in } \mathbb{R}^2 \times (0, \infty) \\ \text{(ii)} \quad T^\lambda = \Theta \qquad \text{on } \mathbb{R}^2 \times \{0\} \end{cases} \tag{19}$$

Using (3) and the scaling properties of the white noise W, we find that

$$v_\lambda(x) = \lambda^{-H} V_\lambda(x) \tag{20}$$

where

$$V_\lambda(x) = (2\pi)^{-1/2} \int_{|k| \geqslant 1} e^{ixk} \psi_\infty^{1/2}(\lambda\, |k|)\, |k|^{-(1/2)-H}\, W(dk) \tag{21}$$

In view of our assumption (15) on H, it follows that the covariance

$$\Gamma_\lambda(x) = \langle V_\lambda(x)\, V_\lambda(0) \rangle = \int_{|k| \geqslant 1} e^{ixk} \psi_\infty(\lambda\, |k|)\, |k|^{-1-2H}\, dk \tag{22}$$

of V_λ is finite and satisfies the assumptions of Lemma A.4 in the appendix, hence, for each $\lambda > 0$, V_λ is a.s. locally Hölder continuous.

In view of the above, going back to (19), it is clear that one needs to choose $\alpha(\lambda)$ and $\beta(\lambda)$ so that

$$\alpha(\lambda) = \beta(\lambda)\, \lambda^H \tag{23}$$

Moreover, since the theory presented here should reduce to the well-known results for the asymptotics of KPP reaction-diffusion equations developed in refs. 12, 17, and 18 if v_λ is either identically zero or bounded, it is necessary to assume that

$$\beta(\lambda) = \lambda \tag{24}$$

But then (23) yields

$$\alpha(\lambda) = \lambda^{1+H}$$

With all the above choices (19) can now be rewritten as

$$\begin{cases} T_t^\lambda - \dfrac{K}{2}(\lambda T_{xx}^\lambda + \lambda^{1+2H} T_{yy}^\lambda) + V_\lambda(x)\, T_y^\lambda + w T_x^\lambda \\ \qquad + \dfrac{K}{\lambda} T^\lambda (T^\lambda - 1) = 0 \qquad \text{in } \mathbb{R}^2 \times (0, \infty) \\ T^\lambda = \Theta \qquad \text{on } \mathbb{R}^2 \times \{0\} \end{cases}$$

Since $0 \leqslant \Theta \leqslant 1$, it is immediate from the maximum principle that

$$0 \leqslant T^\lambda \leqslant \bar{T}^\lambda \qquad \text{in } \mathbb{R}^2 \times [0, \infty) \tag{25}$$

where $\bar{T}^\lambda$ is the solution of the linear initial value problem

$$\begin{cases} \bar{T}_t^\lambda - \frac{\kappa}{2}(\lambda \bar{T}_{xx}^\lambda + \lambda^{1+2H}\bar{T}_{yy}^\lambda) + V_\lambda(x)\,\bar{T}_y^\lambda + w\bar{T}_x^\lambda \\ \qquad -\dfrac{K}{\lambda}\bar{T}^\lambda = 0 \qquad \text{in } \mathbb{R}^2 \times (0, \infty) \\ \bar{T}^\lambda = \Theta \qquad \text{on } \mathbb{R}^2 \times \{0\} \end{cases} \tag{26}$$

Our strategy to prove Theorem 1 is to analyze $\bar{T}^\lambda$ using its Feynman–Kac representation formula and to study the set in which $\bar{T}^\lambda \to 0$. One can, of course, study the whole problem differently by introducing an exponential change of variables $Z^\lambda = \lambda \ln T^\lambda$ and studying the problem satisfied by Z^λ and its limit as $\lambda \to 0$. This is related to the methodology for the usual KPP equation[12, 17, 18] and will be developed in a forthcoming paper.[21]

Returning to the study of (26), it is necessary to understand the a.s. behavior of the Gaussian process V_λ in the limit $\lambda \to 0$. The following proposition is an immediate consequence of (3), the assumptions on ψ_∞, and Propositions A.3 and A.6 in the appendix.

Proposition 1.1. Let V_λ and V be given by (4) and (16), respectively, and assume (15). Then:

(i) As $\lambda \to 0$, $V_\lambda \to V$ a.s. and locally uniformly in $\mathbb{R}$.

(ii) V is a.s. locally Hölder continuous on $\mathbb{R}$. Moreover, for every $\eta > 0$, for almost every ω and $\delta \in (0, H)$ there exists a constant $g_0 = g_0(\eta, \delta, \omega)$ such that for every $x_0 \in \mathbb{R}$ and $R > 0$

$$\sup_{x, y \in [x_0 - R, x_0 + R]} |V(y) - V(x)| \leqslant V_0(1+\eta)[C_H R^\delta + 2(g_0 R)^{\delta - H} \|V\|_{L^\infty(-\bar{R}, \bar{R})}]\,|x-y|^{H-\delta}$$

where $\bar{R} = |x_0| + R$.

We may now proceed with the proof of Theorem 1.

Proof of Theorem 1. 1. Throughout what follows we fix an ω such that Proposition 1.1 holds.

2. The solution $\bar{T}^\lambda$ of (26) is given by the formula

$$\bar{T}^\lambda(x, y, t) = e^{Kt/\lambda}\mathbb{E}\Theta\left(y + (\lambda^{1+2H}\kappa)^{1/2} B_2(t) - \int_0^t V_\lambda(x - ws + (\lambda\kappa)^{1/2} B_1(s))\,ds\right)$$

where $B=(B_1, B_2)$ is a two-dimensional Brownian motion and $\mathbb{E}$ is the expectation with respect to the measure induced by B.

3. For $i=1, 2$ define

$$X_i(t)=\sup_{0\leqslant s\leqslant t}|B_i(s)|$$

It follows from elementary considerations that, for every $\zeta>0$,

$$e^{Kt/\lambda}P((\lambda\kappa)^{1/2}X_1(t)\geqslant(2(1+\zeta)\kappa K)^{1/2}t)=o(1)\qquad\text{as}\quad\lambda\to 0$$

and

$$e^{Kt/\lambda}P((\lambda\kappa)^{(1/2)+H}X_2(t)\geqslant\lambda^{H/4})=o(1)\qquad\text{as}\quad\lambda\to 0$$

4. The last two estimates yield

$$\begin{aligned}\bar{T}^\lambda(x, y, t)\leqslant e^{Kt/\lambda}\mathbb{E}\bigg[\Theta\left(y-\lambda^{H/4}-\int_0^t V_\lambda(x-ws+(\lambda\kappa)^{1/2}B_1(s))\,ds\right)\\ \times\mathbb{1}_{[0,(2(1+\zeta)\kappa K)^{1/2}t]\times[0,\lambda^{H/4}]}((\lambda\kappa)^{1/2}B_1(t),(\lambda\kappa)^{(1/2)+H}B_2(t))\bigg]\\ +o(1)\end{aligned}$$

where $\mathbb{1}_{[a,b]}$ denotes the characteristic function of the interval $[a, b]$.

5. Proposition 1.1(i) yields that if $(\lambda\kappa)^{1/2}X_1(t)\leqslant(2(1+\zeta)\kappa K)^{1/2}t$, then

$$\sup_{0\leqslant s\leqslant t}|(V-V_\lambda)(x-ws+(\lambda\kappa)^{1/2}B_1(s))|=o(1)\qquad\text{as}\quad\lambda\to 0$$

Moreover, Proposition 1.1(ii) implies that, if $(\lambda\kappa)^{1/2}X_1(t)\leqslant(2(1+\zeta)\kappa K)^{1/2}t$, then

$$\sup_{0\leqslant s\leqslant t}|V(x-ws+(\lambda\kappa)^{1/2}B_1(s))-V(x-ws)|\leqslant\bar{\bar{C}}\,|(\lambda\kappa)^{1/2}X_1(t)|^{H-\delta}$$

where

$$\bar{\bar{C}}=V_0(1+\eta)(\bar{C}R_0^\delta+2(R_0\,g_0)^{\delta-H}\,\|V\|_{L^\infty(-R,R)})\tag{27}$$

and

$$R_0=2(1+\zeta)\kappa K)^{1/2}t,\qquad R=|x|+(|w|+(2(1+\zeta)\kappa K)^{1/2})\,t$$

Since Θ is a decreasing function, substituting all the above estimates in the inequality in Step 4 above gives

$$\begin{aligned}
&\bar{T}^{\lambda}(x, y, t) - o(1) \\
&\quad \leqslant e^{Kt/\lambda}\mathbb{E}\left[\Theta\left(y - \int_0^t V(x - ws)\, ds - \lambda^{H/4} - o(1)\, t \right.\right. \\
&\quad\quad \left.\left. - t\bar{C}\, |(\lambda\kappa)^{1/2} X_1(t)|^{H-\delta}\right)\right] \\
&\quad = e^{Kt/\lambda} P\left((\bar{C}t)^{-1}\left(y - \int_0^t V(x - ws)\, ds - \lambda^{H/4}\right.\right. \\
&\quad\quad \left.\left. - o(1)\, t\right) \leqslant (\lambda\kappa)^{1/2}\, |X_1(t)|^{H-\delta}\right)
\end{aligned}$$

In view of the bound which we are trying to achieve, below we pick y large enough so that

$$A = y - \int_0^t V(x - ws)\, ds - \lambda^{H/4} o(1)\, t \geqslant 0$$

Then

$$\bar{T}^{\lambda}(x, y, t) - o(1) \leqslant 2e^{Kt/\lambda} F((\lambda\kappa t)^{1/2}\, ((\bar{C}t)^{-1} A)^{1/(H-\delta)})$$

where

$$F(\beta) = \int_\beta e^{-u^2/2}\, du \tag{28}$$

The elementary fact that

$$F(\beta) \leqslant O\left(\frac{1}{\beta} e^{-\beta^2/2}\right) \qquad \text{for} \quad \beta \gg 1 \tag{29}$$

now yields that the right-hand side of the last inequality converges to 0 as $\lambda \to 0$ if

$$2\kappa K t < ((\bar{C}t)^{-1} A_0)^{2/(H-\delta)}$$

where

$$A_0 = y - \int_0^t V(x - ws)\, ds$$

Rewriting this last inequality gives

$$\bar{C}t[(2\kappa K)^{1/2}\, t]^{H-\delta} < y - \int_0^t V(x - ws)\, ds$$

and, in view of (25) and (27), the claim in Theorem 1. ∎

2. ALMOST SURE UPPER BOUNDS FOR SMOOTH VELOCITY FIELDS

Throughout this section we assume

$$H < 0 \tag{30}$$

and write

$$\ln_+ R = \ln(\max(R, e)) \tag{31}$$

The precise statement of Theorem 2 from the Introduction is:

Theorem 2 (The smooth case). Assume that $H<0$, choose $\alpha(\lambda) = \lambda(|\ln \lambda|)^{1/2}$ and $\beta(\lambda) = \lambda$ in (7), and let

$$\gamma_0 = V_0 \left((2\pi)^{-1} \int \psi_\infty(|k|)\, |k|^{-1-2H}\, dk \right)^{1/2}$$

Then for any $\zeta > 0$ and almost every ω, there exists a constant $C = C(\omega)$ such that, as $\lambda \to 0$, $T^\lambda(x, y, t, \omega) \to 0$, if

$$y > C(\omega)\, \gamma_0 [\ln_+ [\,|x| + (|w| + (2(1+\zeta)\, \kappa K)^{1/2})\, t]]^{1/2}\, t \tag{32}$$

Before we present the proof of Theorem 2, which follows along the same lines as the proof of Theorem 1, we continue with a general discussion motivating the choice of $\alpha(\lambda)$ and $\beta(\lambda)$ above.

In view of (30), it follows that, as $\lambda \to 0$,

$$\Gamma_\lambda(0) \to \gamma_0^2 \tag{33}$$

where γ_0 is defined in the statement of Theorem 2 and

$$\Gamma_\lambda(x) = \langle v_\lambda(x)\, v_\lambda(0) \rangle = (2\pi)^{-1}\, V_0^2 \int_{|k| \geq \lambda} e^{ixk} \psi_\infty(|k|)\, |k|^{-1-2H}\, dk \tag{34}$$

is the covariance of v_λ.

It follows from refs. 22–24—see also Proposition A.1 in the appendix—that, as $\lambda \to 0$, for almost every ω there exists a constant $C(\omega)$ such that

$$\sup_{y \in [-R, R]} |v_\lambda(y)| \leqslant C(\omega)\, \gamma_0 (\ln_+ R)^{1/2} \tag{35}$$

Now going back to (19), which is the equation satisfied by T^λ, and using the same reasoning as in the previous section for the choice of $\beta(\lambda)$, we see that it is appropriate to select

$$\beta(\lambda) = \lambda \qquad \text{and} \qquad \alpha(\lambda) = \lambda(|\ln \lambda|)^{1/2} \tag{36}$$

We proceed now with the proof of Theorem 2.

Proof of Theorem 2. 1. As in the proof of Theorem 1 and with the above choice (36) of $\beta(\lambda)$ and $\alpha(\lambda)$, here we study the a.s. asymptotics of the solution $\bar{T}^\lambda$ of

$$\begin{cases} \bar{T}^\lambda_t - \dfrac{\kappa}{2}(\lambda \bar{T}^\lambda_{xx} + \lambda\, |\ln \lambda|^{1/2}\, \bar{T}^\lambda_{yy}) + |\ln \lambda|^{1/2}\, v_\lambda\left(\dfrac{x}{\lambda}\right) \bar{T}^\lambda_y \\ \qquad + w \bar{T}^\lambda_x - \dfrac{K}{\lambda}\bar{T} = 0 \qquad \text{in } \mathbb{R}^2 \times (0, \infty) \\ \bar{T}^\lambda = \Theta \qquad \text{on } \mathbb{R}^2 \times \{0\} \end{cases} \tag{37}$$

It follows that

$$\bar{T}^\lambda(x, y, t) = e^{Kt/\lambda} \mathbb{E}\Theta \left(y + (\lambda\kappa)^{1/2} B_2(t) \right.$$
$$\left. - |\ln \lambda|^{1/2} \int_0^t v_\lambda \left(\frac{x - ws + (\lambda\kappa)^{1/2} B_1(s)}{\lambda} \right) ds \right)$$

where, as before, $\mathbb{E}$ is the expectation associated with the Brownian motion (B_1, B_2) in $\mathbb{R}^2$.

2. Using the notation of and arguing as in Step 3 of the proof of Theorem 1, we see that, for every $\zeta > 0$,

$$\bar{T}^\lambda(x, y, t) - o(1)$$
$$\leqslant e^{Kt/\lambda}\, \mathbb{E} \left\{ \Theta \left(y - |\ln \lambda|^{1/2} \int_0^t v_\lambda \left(\frac{x - ws + (\lambda\kappa)^{1/2} B_1(s)}{\lambda} \right) ds \right) \right.$$
$$\left. \times\, \mathbb{1}_{[0,\, 2(2(1+\zeta)\kappa K)^{1/2}]}((\lambda\kappa)^{1/2} X_1(t)) \right\}$$

3. Using the inequality in (35) with

$$R = (|x| + (|w| + (2(1+\zeta)\,\kappa K)^{1/2})\,t)$$

in the above upper bound for $\bar{T}^{\lambda}$ yields

$$\bar{T}^{\lambda}(x, y, t) \leqslant e^{Kt/\lambda}\mathbb{E}\Theta(y - t(\ln R)^{1/2}\,C)$$

where

$$C = C(\omega)\,\gamma_0(\ln_+ R)^{1/2}$$

4. It now follows that $\bar{T}^{\lambda} \to 0$, as $\lambda \to 0$, if

$$y > C(\omega)\,\gamma_0(\ln_+[\,|x| + (|w| + (2(1+\zeta)\,\kappa K)^{1/2})\,t\,])^{1/2}\,t$$

hence the claim in (30). ∎

4. ENSEMBLE-AVERAGED UPPER BOUNDS

We begin by restating Theorem 3 from the Introduction, which is about the asymptotic behavior of the ensemble average over all velocity realizations $\langle T^{\lambda} \rangle$, given by (12).

Theorem 3 (Ensemble-averaged bounds). (A) Let $0 < H < 1$ and choose $\alpha(\lambda)$, $\beta(\lambda)$ to satisfy $\beta(\lambda) = \alpha^{2/3}(\lambda)\,\lambda^{-2H/3}$ and $\lambda\beta^{-1}(\lambda) \to 0$ as $\lambda \to 0$. Then, as $\lambda \to 0$, $\langle T^{\lambda} \rangle \to 0$ for

$$y > (2\kappa K C_0)^{1/2}\,t^{3/2} \tag{38}$$

with $C_0 = V_0^2(2\pi)^{-1} \int_{|k| \geqslant 1} |k|^{-1-2H}\,dk$.

(B) For $H < 0$, choose $\alpha(\lambda) = \beta(\lambda) = \lambda$. Then, as $\lambda \to 0$, $\langle T^{\lambda} \rangle \to 0$ for

$$y > (2\kappa K)^{1/2}\,t \tag{39}$$

As in the previous sections, the bounds in (38) and (39) are obtained by analyzing the behavior as $\lambda \to 0$ of the ensemble average $\langle \bar{T}^{\lambda} \rangle$ of the solution $\bar{T}^{\lambda}$ of the linear problem. The latter is accomplished using arguments similar to those introduced in refs. 15, 16, and 19 exploiting the Feynman–Kac representation formula for $\bar{T}^{\lambda}$ and the fact that v_{λ} is a Gaussian random field. As mentioned in the Introduction, results similar to (38) and (39) were obtained independently in ref. 20 with the special choice of $\alpha(\lambda) = \lambda$ and for the general time-dependent velocity fields in refs. 15, 16, and 19.

Before we begin with the proof of Theorem 3, we recall a basic fact about Gaussian random fields, which we state below as a lemma.

Lemma 3.1. Let u be a zero-mean stationary Gaussian process with covariance Γ. If B: $\mathbb{R} \to \mathbb{R}$ is a continuous function, then

$$\left\langle \exp\left[i\xi \int_0^A u(x+B(s))\, ds \right] \right\rangle = \exp\left[-\frac{\xi^2}{2} \int_0^A \int_0^A \Gamma(B(s)-B(s'))\, ds\, ds' \right]$$

We continue with the proof of Theorem 3.

Proof of Theorem 3. 1. Taking the Fourier transform with respect to y, using the Feynman–Kac formula, inverting afterward the Fourier transform, and using Lemma 3.1 yields

$$\begin{aligned} &\langle \bar{T}^\lambda(x, y, t) \rangle \\ &\quad = \left[\exp \frac{Kt}{\beta(\lambda)} \right] \mathbb{E} \int \Theta(\bar{y}) \\ &\qquad \times \int \exp[i(\bar{y}-y)\,\xi] \exp\left\{ -2^{-1} \left[\frac{\alpha^2}{\beta} \kappa t + \frac{\alpha^2}{\beta^2} t^2 Y_\lambda(\beta, t) \right] \xi^2 \right\} d\xi\, d\bar{y} \end{aligned}$$

where $\mathbb{E}$ is the expectation associated with the Brownian motion B on $\mathbb{R}$ and

$$\begin{aligned} Y_\lambda(B, t) = \frac{V_0^2}{2\pi} \int_0^1 \int_0^1 \int_{|k| \geq \lambda} &\exp\left\{ i \left[\left(\frac{t\kappa}{\beta} \right)^{1/2} (B(s)-B(s')) - \frac{t}{\beta} w(s-s') \right] k \right\} \\ &\times \psi_\infty(|k|)\, |k|^{-1-2H}\, dk \end{aligned}$$

2. Assume that $0 < H < 1$. Then

$$\begin{aligned} Y_\lambda(B, t) = \frac{V_0}{2\pi} \lambda^{-2H} \int_0^1 \int_0^1 \int_{|k| \geq 1} &\left(\exp\left\{ i \left[\left(\frac{\lambda^2 t\kappa}{\beta} \right)^{1/2} (B(s)-B(s')) - \frac{\lambda t}{\beta} w(s-s') \right] k \right\} \right. \\ &\left. \times \psi_\infty(\lambda\, |k|)\, |k|^{-1-2H} \right) dk\, ds\, ds' \end{aligned}$$

If β is chosen so that

$$\lambda \beta^{-1}(\lambda) \to 0 \qquad \text{as} \quad \lambda \to 0$$

then

$$Y_\lambda(B, t) = \lambda^{-2H}(C_0 + o(1))$$

where C_0 is defined in the statement of the theorem. Hence

$$\langle \bar{T}^{\lambda}(x, y, t)\rangle = e^{Kt/\beta(\lambda)}(2a_{\lambda})^{-1/2}\, \mathbb{E} \int \Theta(\bar{y})\, e^{-(y-\bar{y})^2/2a_{\lambda}}\, d\bar{y}$$

with

$$a_{\lambda} = \frac{\alpha^2}{\beta} \kappa t + \frac{a^2}{\beta^2(\lambda)} t^2 \lambda^{-2H}(C_0 + o(1))$$

The usual estimate (29) regarding the asymptotics of the exponential now yields in the limit $\lambda \to 0$ that

$$\langle T^{\lambda}\rangle \to 0 \qquad \text{if} \quad y > (2\kappa K C_0)^{1/2}\, t^{3/2}$$

provided that α is chosen so that

$$\alpha^2 = \beta^3 \lambda^{2H}$$

since, with this choice, $\alpha^2/\beta^2 \to 0$ as $\lambda \to 0$.

3. Assume next $H < 0$. Letting $\alpha(\lambda) = \beta(\lambda) = \lambda$, we find that

$$\frac{\alpha^2}{\beta^2} Y_{\lambda}(B, t) = o(1) \qquad \text{as} \quad \lambda \to 0$$

Hence, again from the asymptotics of the tail of the integral of the Gaussian we find that, if $y > (2K\kappa)^{1/2}\, t$, then, as $\lambda \to 0$,

$$\langle T^{\lambda}(x, y, t)\rangle \to 0 \quad \blacksquare$$

4. APPENDIX

We summarize here the facts about stationary Gaussian random fields we used in this paper. For a complete discussion about the properties of such fields see refs. 22–24. For the facts used in this paper for which we could not find exact references, we sketch below some of the steps in their proof.[22–24]

We begin with a result about the growth of Gaussian random fields.

Proposition A.1. Let $X\colon \mathbb{R} \to \mathbb{R}$ be a stationary Gaussian random field with mean zero and covariance $\Gamma(x)$. Assume that almost every realization of X is continuous and that $\Gamma(x) \to 0$ as $|x| \to \infty$. Then

$$\lim_{R \to \infty} \left(\max_{|x| \leqslant R} |X(x)|\right)(2\Gamma(0) \ln R)^{-1/2} = 1 \qquad \text{a.s.}$$

Next assume (15) and consider the Gaussian random field Z^{λ} with spectral representation

$$Z_{\lambda}(x) = (2\pi)^{-1/2} V_0 \int_{|k| \geqslant 1} e^{ixk}[1 - \psi_{\infty}^{1/2}(\lambda |k|)] |k|^{-(1/2)-H} W(dk) \quad \text{(A1)}$$

and denote by $\gamma_{\lambda}(x)$ its covariance. Since

$$\begin{aligned} \gamma_{\lambda}(x) &= \langle Z_{\lambda}(x) Z_{\lambda}(0) \rangle \\ &= (2\pi)^{-1} V_0^2 \int_{|k| \geqslant 1} e^{ixk}[1 - \psi_{\infty}^{1/2}(\lambda |k|)]^2 |k|^{-1-2H} dk \end{aligned} \quad \text{(A2)}$$

it follows from the assumptions on ψ_{∞} that

$$\begin{aligned} (2\pi) |\gamma_{\lambda}(x)| &\leqslant V_0^2 \int_{|k| \geqslant 1} [\psi_{\infty}^{1/2}(0) - \psi_{\infty}^{1/2}(\lambda |k|)]^2 |k|^{-1-2H} dk \\ &\leqslant V_0^2 \, \|(\psi_{\infty}^{1/2})'\|_{\infty} \lambda^2 \int_{k_0}^{\lambda^{-1}} |k|^{1-2H} dk \\ &\quad + 2(1 + \sup_{|x| \geqslant \lambda^{-1}} \psi_{\infty}) \int_{\lambda^{-1}}^{\infty} |k|^{-1-2H} dk \\ &\leqslant V_0^2 \left(\|(\psi_{\infty}^{1/2})'\|_{\infty} \frac{1}{2(1-H)} + 2(1 + \sup_{[\lambda^{-1}, \infty)} \|\psi_{\infty}\|) \frac{1}{2H} \right) \lambda^{2H} \end{aligned}$$

We summarize the above computation by writing

$$\|\gamma_{\lambda}\|_{\infty} \leqslant C_1 \lambda^{2H} \quad \text{(A3)}$$

where C_1 is the constant appearing in the last inequality.

It is also necessary to find an upper bound on the modulus of continuity of the covariance of the Gaussian process Z_{λ}. Since Z_{λ} is stationary, this follows from the following computation:

$$\begin{aligned} &(2\pi)[\gamma_{\lambda}(0) - \gamma_{\lambda}(x)] \\ &\quad = (2\pi)^{-1} V_0^2 \int_{|k| \geqslant 1} (1 - e^{ixk})[1 - \psi_{\infty}^{1/2}(|k|)]^2 |k|^{-1-2H} dk \\ &\quad = (2\pi)^{-1} \cdot V_0^2 \int_{|k| \geqslant 1} (1 - \cos kh)[\psi_{\infty}^{1/2}(0) - \psi_{\infty}^{1/2}(\lambda k)]^2 |k|^{-1-2H} dk \\ &\quad \leqslant (2\pi)^{-1} V_0^2 \left\{ \int (1 - \cos k)[\psi_{\infty}^{1/2}(0) - \psi_{\infty}^{1/2}(\lambda h^{-1} k)]^2 |k|^{-1-2H} dk \right\} h^{2H} \\ &\quad \leqslant \left[(2\pi)^{-1} V_0^2 8 \, \|\psi_{\infty}\|_{\infty} \int_0^{\infty} (1 - \cos k) k^{-1-2H} dk \right] h^{2H} \end{aligned}$$

We record the last estimate as

$$|\gamma_\lambda(x+h)-\gamma_\lambda(x)|\leqslant C_2\sigma^2(h) \tag{A4}$$

where

$$\sigma(h)=h^H \tag{A5}$$

The next lemma is proved in ref. 22.

Lemma A.2. Let X: $[0,1]\to\mathbb{R}$ be a continuous, separable, real-valued Gaussian process with zero mean and continuous covariance $\Gamma(t,s)$. Suppose that

$$E(X(t)-X(s))^2\leqslant\sigma^2(|t-s|)$$

and that $\sigma(h)$ is positive and increasing in h for $h\geqslant 0$. Then for all positive integers n and all $x\geqslant(1+4\log n)^{1/2}$,

$$P\left\{\|X\|_\infty\geqslant x\left((\|\Gamma\|_\infty)^{1/2}+4\int_1^\infty\sigma(n^{-u^2})\,du\right)\right\}\leqslant 4n^2\int_x^\infty e^{-u^2/2}\,du$$

Below we use this lemma and the estimate obtained earlier to establish the following result.

Proposition A.3. Let Z_λ be defined by (A1). Then, as $\lambda\to 0$, $Z_\lambda\to 0$ locally uniformly in x and a.s.

Proof. 1. Here for simplicity we only argue for a sequence $\lambda_n\to 0$.

2. It suffices to show that

$$A_{\lambda_n}=\sup_{x\in[0,1]}|Z_{\lambda_n}(x)|\to 0 \qquad \text{as} \quad \lambda_n\to 0 \qquad \text{a.s. in } \Omega$$

3. Since $P(\|A_{\lambda_n}\|\geqslant a)\leqslant P(\|A_{\lambda_n}\|\geqslant b)$ if $a\geqslant b$, for any $n\in\mathbb{N}$ and $x>(1+4\log n)^{1/2}$, (A3), (A4), and Lemma A.2 yield, for $C=\max(C_1,C_2)$,

$$P\left(A_\lambda\geqslant xC\left(\lambda^H+4\int_1^\infty n^{-Hu^2}\,du\right)\right)\leqslant 4n^2\int_x^\infty e^{-u^2/2}\,du$$

4. Assume for simplicity that $\lambda=n^{-1}$, let $\delta\in(0,H)$, and define x by

$$xC\left(n^{-H}+4\int_1^\infty n^{-Hu^2}\,du\right)=\frac{1}{n^\delta}$$

It is then immediate that, for sufficiently large n,

$$x = n^{H-\delta} \frac{1}{C(1 + 4n^H \int_1^\infty n^{-Hu^2}\, du)} > (1 + 4 \log n)^{1/2}$$

Hence

$$P\left(A_{1/n} \geqslant \frac{1}{n^\delta}\right) \leqslant 4n^2 \int_{n^{H-\delta}[C(1+4n^H \int_1^\infty n^{-Hu^2}\, du)]^{-1}}^{\infty} e^{-u^2/2}\, du$$

It is now a simple exercise to check that

$$\sum_n P\left(A_{1/n} \geqslant \frac{1}{n^\delta}\right) < \infty$$

which, in view of the Borel–Cantelli lemma, yields that, for all but finitely many n's,

$$A_{1/n} \leqslant \frac{1}{n^\delta} \qquad \text{a.s. in } \Omega$$

5. The proof is now complete. ▮

The next result we present here is about the Hölder continuity of the process V defined by (16). The a.s. continuity properties of Gaussian random fields have been the object of extensive investigations. Here we recall a basic result from refs. 22–24 about the Hölder continuity of Gaussian random fields and then show how it can be modified for the purposes needed in this paper.

Lemma A.4. Let $X(t)$ be a real-valued, continuous, separable Gaussian process. Assume that $\mathbb{E}\{(X(t+h) - X(t))^2\} \leqslant \sigma^2(h)$, where $\sigma^2(h)$ is assumed to be concave in $[0, \delta]$ for some $\delta > 0$, and that $\sigma(h)(h\sigma'(h))^{-1} = o(\log h^{-1})$. Then, for all x,

$$\limsup_{\substack{|t-t'|=h \to 0 \\ t, t' \in [-1, 1]}} \frac{|X(x+t) - X(x+t')|}{[2\sigma^2(h) \log(1/h)]^{1/2}} \leqslant 1 \qquad \text{a.s.}$$

We apply Lemma A.4 to the stationary Gaussian random field V defined by (16)

$$V(x) = (2\pi)^{-1/2}\, V_0 \int_{|k| \geqslant 1} e^{ikx}\, |k|^{-(1/2)-H}\, W(dk)$$

which has covariance

$$\Gamma(x) = (2\pi)^{-1} V_0^2 \int_{|k| \geq 1} e^{ikx} |k|^{-1-2H} dk$$

To this end, let $x, y \in [x_0 - R, x_0 + R]$ for some $R > 0$ and $x_0 \in \mathbb{R}$. Then

$$\begin{aligned} V(x_0 + x) - V(x_0 + y) &= V\left(R\left(\frac{x + x_0}{R}\right)\right) - V\left(R\left(\frac{y + x_0}{R}\right)\right) \\ &= \bar{V}\left(\frac{x}{R}\right) - \bar{V}\left(\frac{y}{R}\right) \end{aligned}$$

where $\bar{V}$ is a stationary Gaussian random field, which satisfies, as can be seen after a straightforward calculation similar to the one earlier in this appendix, the hypotheses of Lemma A.4 with

$$\sigma^2(h) = C^2(Rh)^{2H}$$

We use the above inequalities to state the next proposition.

Proposition A.5. Let V be given by (16). For any $\eta > 0$ and for a.s. ω and every $R > 0$ there exists $g_0 = g_0(\eta, \omega)$ and $\bar{C} = \bar{C}(C_H, |x|)$ such that if $x, y \in R$ and $|x - y| \leq Rg_0$,

$$|V(x + x_0) - V(y + x_0)| \leq (1 + \eta)\, \bar{C}(2\, |x - y|^{2H} \log R/|x - y|)^{1/2}$$

It also follows from Lemma A.4 applied to $\bar{V}$ that given $\eta > 0$ and $\delta > 0$ for a.s. each ω in Ω there exists $g_0 = g_0(\eta, \delta, \omega)$ such that

$$|\bar{V}(x) - \bar{V}(y)| \leq (1 + \eta)(2\bar{C}^2)^{1/2} R^H |x - y|^{H-\delta} \qquad \text{if} \quad |x - y| < g_0$$

This last inequality can be easily extended for $|x - y| > g_0$ by

$$|\bar{V}(x) - \bar{V}(y)| \leq [(1 + \eta)(2\bar{C}^2)^{1/2} R^H + g_0^{\delta - H} 2\, \|\bar{V}\|\, L^\infty(-1, 1)]\, |x - y|^{H-\delta}$$

We rewrite this last inequality in terms of the original process V and state the conclusion as the following result.

Proposition A.6. For each $\eta > 0$ and $0 < \delta < H$ and almost every ω there exists $g_0 = g_0(\eta, \delta, \omega)$ such that for all x_0 and $R > 0$

$$\begin{aligned} &\sup_{x, y \in [-R, R]} |V(x + x_0) - V(y + x_0)| \\ &\quad \leq (1 + \eta)[(2\bar{C}^2)^{1/2} R^\delta \\ &\qquad + 2(g_0 R)^{\delta - H} \|V\|_{L^\infty[-(R + |x_0|), R + |x_0|]}]\, |x - y|^{H-\delta} \end{aligned}$$

ACKNOWLEDGMENTS

A.M. is partially supported by grants from the National Science Foundation, the Office of Naval Research, and the Army Research Office. P.E.S. is partially supported by grants from the National Science Foundation, the Office of Naval Research, and the Army Research Office.

REFERENCES

1. R. G. Abdel-Gayed, K. J. Al-Khishali, and D. Bradley, Turbulent burning velocities and flame straining in explosions, *Proc. R. Soc. Lond. A* **391**:394–414 (1984).
2. R. G. Abdel-Gayed, D. Bradley, M. N. Hamid, and M. Lawes, Lewis number effects on turbulent burning velocity, in *Twentieth Symposium (International) on Combustion* (Combustion Institute, 1984), pp. 505–512.
3. W. T. Ashurst and G. I. Sivashinsky, On flame propagation through periodic flow fields, *Combust. Sci. Technol.* **80**:159–164 (1991).
4. A. Kerstein, W. T. Ashurst, and F. A. Williams, Field equation for interface propagation in an insteady homogeneous flow field, *Phys. Rev. A* **37**:2728–2731 (1988).
5. A. R. Kerstein and W. T. Ashurst, Propagation rate of growing interfaces in stirred fluids, *Phys. Rev. Lett.* **68**:934–937 (1992).
6. N. Peters, A spectral closure for premixed turbulent combustion in the flamelet regime, *J. Fluid Mech.* **242**:611–629 (1992).
7. G. I. Sivashinksy, Cascade renormalization theory of turbulent flame speed, *Combust. Sci. Technol.* **62**:77–96 (1988).
8. V. Yakhot, Propagation velocity of premixed turbulent flames, *Combust. Sci. Technol.* **60**:191–214 (1988).
9. P. Clavin and F. A. Williams, Theory of premixed-flame propagation in large-scale turbulence, *J. Fluid Mech.* **90**:589–604 (1979).
10. P. Clavin and F. A. Williams, Effects of molecular diffusion and of thermal expansion of the structure and dynamics of premixed flames in turbulent flows of large and low intensity, *J. Fluid Mech.* **116**:251–282 (1982).
11. M. Matalon and B. J. Matkowsky, Flames as gas dynamics discontinuities, *J. Fluid Mech.* **124**:239–259 (1982).
12. A. J. Majda and P. E. Souganidis, Large scale front dynamics for turbulent reaction-diffusion equations with separated velocity scales, *Nonlinearity* **7**:1–30 (1994).
13. P. F. Embid, A. J. Majda, and P. E. Souganidis, Effect geometric front dynamics for premixed turbulent combustion with separated velocity scales, *Combust. Sci. Technol.* **103**:85–100 (1994).
14. P. F. Embid, A. J. Majda, and P. E. Souganidis, Comparison of turbulent flame speeds from complete averaging and the *G*-equation, *Phys. Fluids* **7**:2052–2060 (1995).
15. M. Avellaneda and A. J. Majda, Mathematical models with exact renormalization for turbulent transport, *Commun. Math. Phys.* **131**:381–429 (1990).
16. M. Avellaneda and A. J. Majda, Mathematical models with exact renormalization for the turbulent transport, II: Fractal interfaces, Non-Gaussian statistics and the sweeping effect, *Commun. Math. Phys.* **146**:139–204 (1992).
17. M. Freidlin, Limit theorems for large deviations of reaction-diffusion equations, *Ann. Prob.* **13**:639–675 (1985).
18. L. C. Evans and P. E. Souganidis, A PDE approach to geometric optics for certain semilinear parabolic equations, *Indiana U. Math. J.* **38**:141–172 (1989).

19. A. Majda, Explicit inertial range renormalization theory in a model for turbulent diffusion, *J. Stat. Phys.* 73:515–542 (1993).
20. S. P. Fedotov, Renormalization for reaction front propagation in a fully developed turbulent shear flow, *Phys. Rev. E* (1995), to appear.
21. A. Majda and P. E. Souganidis, Lower bounds on enhanced turbulent flame speeds for combustion with fractal velocity fields, In preparation.
22. X. Fernique, Regularité des fonctions aleatoires gaussiennes, in *Lecture Notes in Mathematics*, Vol. 480 (Springer-Verlag, Berlin), pp. 1–91.
23. X. Fernique, Regularité des fonctions aleatoires nongaussiennes, in *Lecture Notes in Mathematics*, Vol. 976 (Springer-Verlag, Berlin), pp. 1–69.
24. M. Weber, Analyse infinitesimale des fonctions aleatoires, *Lecture Notes in Mathematics*, Vol. 976 (Springer-Verlag, Berlin), pp. 383–465.